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A Comparison between Adomian's Decomposition Methods and Perturbation Techniques for Nonlinear Random Differential Equations

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Comparison of the Adomian decomposition method and regular perturbation techniques applied to the solution of nonlinear vector random differential equations shows that the decomposition method is superior and generally applicable. Even for the cases where perturbation is applicable, the comparison shows that the decomposition method is easier to compute and supplies quantitatively reliable results.
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1. INTRODUCTION

Consider the class of mathematical models of real physical systems defined by nonlinear vector ordinary differential equations with random parameters written in the operator form[1]

$$i = 1,..., n: \qquad Lx_i(\omega, t) = N_i(\mathbf{x}(\omega, t), \mathbf{r}(\omega, t), t; \varepsilon) \tag{1}$$

with initial conditions $x_i(t=0) = x_{i0}$.

In Eq. (1), $\mathbf{x} = \{x_i^T\}: \Omega \cdot I \to \mathbb{D} \subseteq \mathbb{R}^n$ is the random state variable defining the "physical state" of the real system modelled by the equation, $I = [0, t] \in \mathbb{R}$ is the domain of the independent variable t, Ω is an abstract space of the elementary events ω defined in a complete probability space $(\Omega, \mathfrak{B}, p)$, $\mathbf{r}(\omega, t)$ is a set of known bounded stochastic processes, and finally ε is a deterministic parameter characterizing the system. Moreover, in the operator formulation of Eq. (1), L is the ordinary differential operator d/dt and $N = \{N_i^T\}$ is a set of n nonlinear functions of $\mathbf{x}$, $\mathbf{r}$; t and ε are analytical in all these arguments.

[1] The method has very recently been extended to algebraic systems, nonlinear and/or stochastic partial differential equations, delay differential equations, and systems of equations. We have dealt here only with differential equations.

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Considering that the functions N_i are nonlinear and that no "a priori" assumption is made upon the structure of $\mathbf{r}$ (such as white noise hypotheses, etc.), the actual solution of Eq. (1) can be a hard problem to tackle. On the other hand, a mathematical model has to simulate nature, however difficult; whereas nature usually cannot be cast into an equation which is easy to solve.

Adomian's decomposition method, well documented in his recent book [1] and related bibliography, can be successfully applied towards this objective. On the other hand, if ε is a small parameter and some further conditions, afterwards specified, are fulfilled, some regular perturbation techniques [2], extended to random differential equations [3], can also be applied. A comparison between the two methods is then desirable.

A comparison is realized in the following section after a brief review of the two methods and in the application considered in the concluding paragraph. This analysis clearly indicates that the decomposition method is far more tractable and computable, and requires fewer conditions in the structure of Eq. (1), and in particular, on the nonlinear operators N_i, than the regular perturbation techniques which are founded on the rather strong hypothesis of smallness of the parameter ε. This conclusion holds also for quantitative results.

2. Methods and Qualitative Comparison

Let us first discuss, referring to Eq. (1), the solution method via the perturbation technique. According to this method, the solution is sought in the form

$$\mathbf{x} = \mathbf{x}(t, \omega; \mathbf{x}_0) = \sum_{h=0}^{m} \varepsilon^h \mathbf{x}^{(h)}(t, \omega; \mathbf{x}_0), \tag{2}$$

the functions $\mathbf{N}$ being expanded in powers of ε as

$$\mathbf{N} \cong \sum_{h=0}^{m} (1/h!)(d^h \mathbf{N}/d\varepsilon^h)_{\varepsilon=0}\, \varepsilon^h = \sum_{h=0}^{m} \mathbf{N}^{(h)} \varepsilon^h; \tag{3}$$

as known,

$$\begin{aligned}
&\mathbf{N}^{(0)} = \mathbf{N}(\mathbf{x}^{(0)}, \mathbf{r}, t; \varepsilon = 0) \\
&\mathbf{N}^{(1)} = [a_{ij}(\mathbf{x}^{(0)}; \mathbf{r}, t)]\, \mathbf{x}^{(1)} + \mathbf{b}_1(\mathbf{x}^{(0)}, \mathbf{r}, t) \\
&\quad\vdots \\
&\mathbf{N}^{(m)} = [a_{ij}(\mathbf{x}^{(0)}; \mathbf{r}, t)]\, \mathbf{x}^{(m)} + (1/m!)\, \mathbf{b}_m(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m-1)}, \mathbf{r}, t),
\end{aligned} \tag{4}$$

where $a_{ij} = (\partial N_i/\partial x_j)_{\varepsilon=0}$ and $\mathbf{b}_m = \{b_{im}^T\}$ with

$$b_{im} = (d^m N_i/d\varepsilon^m)_{\varepsilon=0} - m! \sum_{j=1}^{n} (\partial N_i/\partial x_j)_{\varepsilon=0} (d^m x_j/d\varepsilon^m)_{\varepsilon=0}. \tag{5}$$

Therefore if Eqs. (2)–(4) are cast into the form of Eq. (1) and the terms with equal powers of ε are equated, the following sequence is obtained:

$$L\mathbf{x}^{(0)} = \mathbf{N}^{(0)}(\mathbf{x}^{(0)}, \mathbf{r}, t; \varepsilon = 0), \qquad \mathbf{x}_0^{(0)} = \mathbf{x}_0 \tag{6a}$$

$$L\mathbf{x}^{(1)} = [a_{ij}(\mathbf{x}^{(0)}, \mathbf{r}, t)]\,\mathbf{x}^{(1)} + \mathbf{b}_1(\mathbf{x}^{(0)}, \mathbf{r}, t), \qquad \mathbf{x}_0^{(1)} = \mathbf{0} \tag{6b}$$

$$\vdots$$

$$L\mathbf{x}^{(m)} = [a_{ij}(\mathbf{x}^{(0)}, \mathbf{r}, t)]\,\mathbf{x}^{(m)} + \mathbf{b}_m(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(m-1)}, \mathbf{r}, t). \qquad \mathbf{x}_0^{(m)} = \mathbf{0} \tag{6c}$$

Then, if the first equation can be solved in some time interval and $\mathbf{x}^{(0)}(\omega, t)$ is its solution, Eqs. (6b) and (6c) which follow define a sequence of linear random differential equations, whose solution can be put in the form

$$\mathbf{x}^{(m)} = [G(t, \mathbf{r})] \int_0^t [G^{-1}(t, \mathbf{r})]\,\mathbf{b}_m(\omega, s, \mathbf{r})\,ds, \tag{7}$$

where $[G]$ is the transition matrix of the linear homogeneous equation $L\mathbf{x}^{(m)} = [a_{ij}]\,\mathbf{x}^{(m)}$.

On the other hand, Adomian's decomposition method provides the solution in terms of decomposition of both the solution and the nonlinear term appearing in Eq. (1). Considering that the operator L is invertible, Eq. (1) can be written as

$$x_i = x_{0i} + L^{-1}N_i(\mathbf{x}, \mathbf{r}, t; \varepsilon), \tag{8}$$

where $\mathbf{x}$ is decomposed as

$$\mathbf{x} \cong \mathbf{x}^0 + \sum_{h=1}^{m} \lambda^h \mathbf{x}^{(h)}, \qquad \lambda = 1, \tag{9}$$

and analogously the term $L^{-1}N_i$ is decomposed as

$$L^{-1}N_i = \mathfrak{F}_i = \sum_{h=1}^{m} \lambda^h \mathfrak{F}_i^{(h)}, \tag{10}$$

where

$$\mathfrak{F}_i^{(h)} = (1/h!)(d^h \mathfrak{F}_i/d\lambda^h)_{\lambda=0} = (1/h!)\,L^{-1}(d^h N_i/d\lambda^h)_{\lambda=0}. \tag{11}$$

Then, after Eq. (11), $\mathfrak{F}_i^{(h)} = \mathfrak{F}_i^{(h)}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(h)})$. Moreover equating terms with the same order of decomposition (as λ is not a small parameter, but an

artificial "decomposition" parameter) the following sequence of quadratures is obtained:

$$x_i^{(0)} = x_{i0} \tag{12a}$$

$$x_i^{(1)} = \mathfrak{F}_i^{(0)}(\mathbf{x}^{(0)}) \tag{12b}$$

$$\vdots$$

$$x_i^{(m)} = \mathfrak{F}_i^{(m-1)}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(m-1)}). \tag{12c}$$

After this brief review of the two methods, a comparison can be realized. Since the final objective of the two methods is the solution of problems or, in other words, obtaining quantitative results for mathematical models of real physical systems, the comparison has to consider whether the mathematical method can reach this objective by a sufficiently simple computation. Keeping this in mind, the following remarks can be stated:

Remark I. The decomposition method does not require that a "small" parameter exists in Eq. (1) nor that additionally the equation $\dot{x} = N(x, \mathbf{r}; \varepsilon = 0)$ has an analytical solution. On the other hand, the perturbation method requires both these strong conditions.

Remark II. The decomposition method is much easier to compute, as it involves a sequence of quadratures of terms, which are easy to compute. On the other hand, the perturbation method involves a sequence of linear differential equations, each of which, in the framework of random differential equations, can be as difficult to solve as the original nonlinear equation.

Remark III. In the framework of stochastic equations, the decomposition method allows in a much easier form the computation of the moments, whereas the perturbation method implies a very cumbersome computation of the moments beginning from the zero-order equation. In fact in the first method the solution is obtained by quadrature involving also the stochastic process, and in the second method the solution is obtained after the treatment of a sequence of linear differential equations involving a stochastic process.

Of course, various particular cases can be found such that the perturbation method can be easy to apply. This is the case with the semilinear "weakly" perturbed equations

$$Lx_i = g_i(t) + f_i(\varepsilon, t)\, N_i(\mathbf{x}, t, \mathbf{r}), \qquad f_i(\varepsilon = 0, t) = 0. \tag{13}$$

However, Eq. (13) has to be considered a very particular and "lucky" mathematical model.

As far as the problem of the convergence to the exact solution is concerned it has to be studied for each particular case, namely, when the structure of the operator N is specified.

3. An Experiment and Conclusions

As an experiment for further comparison between the two methods, outlined in the preceding section, the following equation is considered:

$$Lx = (x+\alpha)^2 \sin(\varepsilon t), \qquad \alpha > 0,\ x_0 = x(t=0) = 0 \tag{14}$$

with α a constant random variable.

The choice of this particular application has been made for the following reasons:

(i) The analytical solution of Eq. (14) can easily be recovered:

$$x_{\text{anal}} = \frac{\alpha(1-\cos(\varepsilon t)}{\varepsilon/\alpha - 1 + \cos(\varepsilon t)}; \qquad t < t_m = \frac{1}{\varepsilon}\cos^{-1}(1-\varepsilon/\alpha), \tag{15}$$

allowing a comparison between analytical approximated solutions and the exact one without the uncertainty of the numerical integrations.

(ii) All requirements necessary to apply the perturbation techniques are satisfied by Eq. (14). Moreover Eq. (14) is equivalent to Eq. (13) as $\varepsilon = 0 \Rightarrow Lx = 0$.

(iii) Equation (14) can be a hard test for approximated methods as $t \to t_m \Rightarrow |x| \to \infty$.

The application of the procedure indicated by Eqs. (5) and (11) gives, for the first five terms, the following sequence:

Perturbation method:

$$\begin{aligned}
Lx^{(0)} &= 0\\
Lx^{(1)} &= t(x^{(0)}+\alpha)^2\\
Lx^{(2)} &= 2t(x^{(0)}+\alpha)\,x^{(1)}\\
Lx^{(3)} &= t(x^{(1)})^2 + 2t(x^{(0)}+\alpha)\,x^{(2)} - \tfrac{1}{6}t^3(x^{(0)}+\alpha)^2\\
Lx^{(4)} &= 2tx^{(1)}x^{(2)} + 2t(x^{(0)}+\alpha)\,x^{(3)} - \tfrac{2}{3}t^3(x^{(0)}+\alpha)\,x^{(1)}
\end{aligned}$$

Decomposition method:

$$\begin{aligned}
x^{(0)} &= x_0\\
x^{(1)} &= L^{-1}\{\sin(\varepsilon t)(x^{(0)}+\alpha)\}^2\\
x^{(2)} &= L^{-1}\{2\sin(\varepsilon t)(x^{(0)}+\alpha)\,x^{(1)}\}\\
x^{(3)} &= L^{-1}\{\sin(\varepsilon t)[2x^{(2)}(x^{(0)}+\alpha) + (x^{(1)})^2]\}\\
x^{(4)} &= L^{-1}\{2\sin(\varepsilon t)[x^{(1)}x^{(2)} + x^{(3)}(x^{(0)}+\alpha)]\}
\end{aligned}$$

Then if x_{pert} is the approximated solution obtained by the first method and x_{dec} is the one obtained by the second method, the following result is found:

$$x_{\text{pert}} = \alpha^2 \frac{t^2}{2} + \alpha^3 \frac{t^4}{4} + \frac{1}{24}\alpha^2 t^4 (3\alpha^2 t^2 - 1) + \frac{1}{8}\alpha^3 t^6 \left(\frac{\alpha^2 t^2}{2} - \frac{5}{9}\right)$$

$$x_{\text{dec}} = \frac{\alpha^2}{\varepsilon}[1 - \cos(\varepsilon t)] + \frac{\alpha^3}{\varepsilon^2}\left[\frac{3}{2} - 2\cos(\varepsilon t) + \frac{1}{2}\cos(2\varepsilon t)\right]$$
$$+ \frac{\alpha^4}{\varepsilon^3}\left[\frac{5}{2} - \frac{7}{2}\cos(\varepsilon t) + \frac{3}{2}\cos(2\varepsilon t) - \frac{1}{6}\cos(3\varepsilon t) - \frac{1}{3}\cos^3(\varepsilon t)\right]$$
$$+ \frac{\alpha^5}{\varepsilon^4}\left[\frac{69}{16} - 6\cos(\varepsilon t) + \frac{41}{12}\cos(2\varepsilon t) - \frac{2}{3}\cos(3\varepsilon t) + \frac{5}{48}\cos(4\varepsilon t)\right.$$
$$\left. - \frac{4}{3}\cos^3(\varepsilon t) + \frac{1}{6}\cos^4(\varepsilon t)\right].$$

Comparisons between the approximated and the exact solutions are shown in Tables I and II for fixed values of α. Table I refers to small values of ε ($\varepsilon = 0.1$) and Table II to large values of ε ($\varepsilon = 1$).

Tables I and II clearly indicate how the decomposition method supplies reliable results much closer to the exact ones than the results obtained via the perturbation technique. This is so in spite of the fact that Eq. (13) has been chosen in favour of the perturbation techniques since all conditions necessary for effective perturbation are fulfilled. In fact Eq. (13) belongs to

TABLE I

t	x_{anal}	x_{pert}	x_{dec}
0.000000	0.000000	0.000000	0.000000
0.250000	0.003135	0.003135	0.003138
0.500000	0.012656	0.012655	0.012657
0.750000	0.028925	0.028920	0.028924
1.000000	0.052585	0.052578	0.052582
1.250000	0.084626	0.084690	0.084624
1.500000	0.126493	0.127018	0.126474
1.750000	0.180268	0.182574	0.180165
2.000000	0.248961	0.256578	0.248573
2.250000	0.337004	0.357982	0.335640
2.500000	0.451117	0.501777	0.446906
2.750000	0.601917	0.712299	0.589918
3.000000	0.807126	1.027800	0.775010

TABLE II

t	x_{anal}	x_{pert}	x_{dec}
0.000000	0.000000	0.000000	0.000000
0.100000	0.005021	0.005020	0.005021
0.200000	0.020339	0.020307	0.020339
0.300000	0.046752	0.046406	0.046751
0.400000	0.085704	0.083897	0.085701
0.500000	0.139494	0.133247	0.139463
0.600000	0.211628	0.195310	0.211431
0.700000	0.307459	0.273190	0.306519
0.800000	0.435324	0.376352	0.431641
0.900000	0.608726	0.528106	0.596247
1.000000	0.850816	0.777778	0.812821

the very "lucky" class of equations such that, if the perturbation method is applied, a sequence of quadratures is obtained.

In the decomposition method the advantages always hold. The decomposition method also has the further advantage of separating the random term.

Table III (where now $\varepsilon = 0.5$) shows how, via the decomposition method, the distance $(x_{\text{anal}} - x_{\text{dec}})$ decreases monotonically and for all t, as new terms are added. This is so even for large values of t and near the very crucial condition stated at the point (iii) $(t \to t_m \Rightarrow |x| \to \infty)$. This convergence property is fulfilled by the perturbation terms only for very small

TABLE III

t	x_{anal}	$x_{\text{dec}}^{(1)}$ [a]	$x_{\text{dec}}^{(2)}$	$x_{\text{dec}}^{(3)}$	$x_{\text{dec}}^{(4)}$
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.100000	0.002506	0.002499	0.002506	0.002506	0.002506
0.200000	0.010093	0.009992	0.010092	0.010092	0.010093
0.300000	0.022974	0.022458	0.022962	0.022974	0.022974
0.400000	0.041522	0.039867	0.041456	0.041520	0.041522
0.500000	0.066297	0.062175	0.066041	0.066281	0.066296
0.600000	0.098089	0.089327	0.097306	0.098019	0.098083
0.700000	0.137986	0.121255	0.135957	0.137740	0.137956
0.800000	0.187476	0.157878	0.182803	0.186739	0.187360
0.900000	0.248604	0.199106	0.238749	0.246642	0.248214
1.000000	0.324214	0.244835	0.304779	0.319455	0.323049
1.100000	0.418341	0.294951	0.381947	0.407607	0.415175
1.200000	0.536874	0.349329	0.471359	0.513988	0.528880
1.300000	0.688711	0.407832	0.574160	0.641993	0.669658
1.400000	0.887917	0.470316	0.691512	0.795545	0.844473

[a] $x_{\text{dec}}^{(i)}$ is the decomposition method solution expanded to terms of i order.

values of t. In fact for large t the addition of new terms does not improve the solution.

As a further example consider the equation

$$Lx = (x+\alpha)^2 \sin(\varepsilon t) + x^2, \qquad x(t=0) = 1; \tag{16}$$

application of the sequence given by Eqs. (4) and (5) is very cumbersome work, even for the first terms (the same is true with Picard's successive approximation method), whereas the sequence of Eq. (11) is definitively easy to compute and has an additional important advantage of separation of the random term. In fact the first three terms with both methods give the result

$$x^{(0)}_{\text{pert}} = \frac{1}{1-t}$$

$$x^{(1)}_{\text{pert}} = \frac{1}{2}\frac{1}{(1-t)^4} + \frac{\alpha}{(1-t)^3} - \frac{3}{2}\frac{\alpha}{(1-t)^2} = \psi(t, \alpha)$$

$$x^{(2)}_{\text{pert}} = \frac{1}{(1-t)^4}\int_0^t \frac{1}{(1-\sigma)^4}\left[\frac{1}{2}\left(\alpha + \frac{1}{1-\sigma}\right)^2 + \psi^2(\sigma, \alpha)\right] d\sigma = \phi(t, \alpha)$$

$$x^{(0)}_{\text{dec}} = 1$$

$$x^{(1)}_{\text{dec}} = t + \frac{1}{\varepsilon}(1+\alpha)^2 (1 - \cos(\varepsilon t))$$

$$x^{(2)}_{\text{dec}} = t^2 + \frac{2}{\varepsilon}(1+\alpha)^2 (1 - t\cos(\varepsilon t)) - \frac{1}{\varepsilon^2}(1+\alpha)^3 (2\cos(\varepsilon t) - \cos^2(\varepsilon t) - 1).$$

This example clearly indicates how additional terms in the decomposition method are easily obtained. On the other hand, continuing the perturbation method becomes quite tedious. This feature is further emphasized in the case of vector differential equations such as the ones considered in Section 2 and when α is a stochastic process.

In addition, from the various comments and results already given concerning the advantages of the decomposition method, one can firmly state that Adomian's method is extremely efficient in supplying analytical solutions in nonlinear models in applied science.

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